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Improvement on the WKB method using an algebraic formalism

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Abstract. The equation $\ddot{u}+w^2(t)u=0$ can be transformed to a system of two first-order differential equations $\dot{y}=My$ by different ways of parametrisation. The method of decoupled exponents of Wei and Norman yields a Riccati differential equation for the unknown exponents and an easily interpretable solution y. A solution with vanishing exponents in the parametrisation of Mulholland corresponds exactly to the WKB method. Therefore an approximate solution of the Riccati differential equation gives an improvement on the WKB.

1. Introduction

Most of the linear differential equations (DE) in physics are of second order. In this paper we shall study an important subset of them, those DE which can be reduced to normal form (for the exact conditions, cf Kamke 1942)

$$\ddot{u}+w^2(t)u=0, \tag{1}$$

or, after introducing a new variable by $u=\exp(\int v\ \mathrm{d}t)$, to the equivalent Riccati DE

$$\dot{v}+v^2+w^2(t)=0. \tag{2}$$

We assume in the following that $w(t)$ is continuous in an interval I_t containing $t=0$; then there exists a global and unique solution for given initial values $u(0)$, $\dot{u}(0)$. Unfortunately, only little is known beyond this statement. Only a few types of DE (1) allow a complete solution; some types have a theory of their own, but, in general, it is impossible to represent the solution by a finite number of quadratures and elementary functions. Therefore one has to have recourse to approximate methods of solution, for example to the WKB method. The condition for the applicability of the WKB method is a coefficient w^2 of the form $w^2(t)=F^2(t)/\lambda^2$ with a large factor $1/\lambda$. The solution of (1) is expressed as an exponential $u=\exp(\mathrm{i}S(t)/\lambda)$, where $S(t)$ is expanded in a power series in λ. The lowest-order solution then reads $u(t)=\exp(\pm\mathrm{i}\int w\ \mathrm{d}t)$. Instead of going into details, we refer to two standard references (Morse and Feshbach 1953, Fröman and Fröman 1965). In § 2 we transform equation (1) into a form suitable for the following investigations by introducing the equivalent matrix DE for the integral matrix Q (fundamental solution matrix) and by extracting two simple factors from Q.

Some years ago, Wei and Norman (1963, 1964) proposed a solution of equation (1) based on Lie algebraic methods. This method cannot, of course, solve equations otherwise unsolvable; the solution is reduced to the solution of a certain Riccati DE.

However, the solution appears in a very appealing form, and it allows an easy improvement on the WKB method. As far as we know, this method has not found many applications in the literature—frequency modulation (Mulholland 1970, Mulholland and Machiraju 1973), light propagation (Strezh 1976), quantum mechanics (Chang and Light 1969, Flamand 1966)—presumably because Lie algebra is not sufficiently well known to physicists and engineers. Therefore we present in § 3 the method of decoupled exponentials of Wei and Norman in a form which uses only simple 2×2 matrix manipulations, thus avoiding the Lie algebraic language. In the main part, § 4, we discuss the connection with the WKB method. In § 5 we give an example for the improvement on the WKB method by a perturbative solution of the Wei–Norman equations. In the Appendix some results from elementary matrix algebra are listed.

In the following, all 'vectors' are 2×1 matrices, all 'matrices' are 2×2 matrices.

2. Preparation of the differential equation

2.1. *The* DE *for the integral matrix* $Q(t)$

Equation (1) can be put in the form of two coupled first-order DE

$$\dot{y} = M(t)y, \tag{3}$$

where the components of

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{4}$$

are some linear combinations of u and $\dot{u}$, and the components of

$$M = \begin{pmatrix} m_1(t) & m_2(t) \\ m_3(t) & m_4(t) \end{pmatrix} \tag{5}$$

depend on $w(t)$. Now the *ansatz*

$$y(t) = Q(t)y(0), \tag{6}$$

where

$$y(0) = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \rho_0 \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix}, \tag{7}$$

transforms equation (3) to the matrix DE for the integral matrix Q:

$$\dot{Q}(t) = M(t)Q(t), \tag{8}$$

with the initial condition

$$Q(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0. \tag{9}$$

2.2. *Three different parametrisations for* y

We choose as a first parametrisation

$$y \to y_{\mathrm{I}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathrm{I}} = \begin{pmatrix} \dot{u} \\ u \end{pmatrix}. \tag{10}$$

Then

$$M \to M_{\mathrm{I}} = \begin{pmatrix} 0 & -w^2 \\ 1 & 0 \end{pmatrix}. \tag{11}$$

One can get a whole class of admissible parametrisations by applying a Lyapunov transformation (Pease 1965) to y_{I}:

$$y_{\mathrm{I}} = B_i(t) y_i \qquad (i = \mathrm{II}, \mathrm{III}, \ldots). \tag{12}$$

Here $B(t)$ is continuously differentiable, $B(t)$ and $\dot{B}(t)$ are bounded, and the magnitude of $\det B(t)$ has a lower bound, in the interval I_t. Substituting this t-dependent coordinate transformation (12) into equation (3) (with index I), we obtain

$$\dot{y}_i = M_i(t) y_i, \qquad \text{with } M_i = B_i^{-1} M_{\mathrm{I}} B_i - B_i^{-1} \dot{B}_i. \tag{13}$$

A Lyapunov transformation transforms Q as follows:

$$Q_{\mathrm{I}}(t) = B_i(t) Q_i(t) B_i^{-1}(0). \tag{14}$$

$i = \mathrm{II}$: With

$$B_{\mathrm{II}} = \begin{pmatrix} \mathrm{i}w & -\mathrm{i}w \\ 1 & 1 \end{pmatrix} \tag{15}$$

(Van Kampen 1967) we get

$$y_{\mathrm{II}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathrm{II}} = \tfrac{1}{2} \begin{pmatrix} u - \mathrm{i}\dot{u}/w \\ u + \mathrm{i}\dot{u}/w \end{pmatrix} \tag{16}$$

and

$$M_{\mathrm{II}} = \mathrm{i}w \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\dot{w}}{2w} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{17}$$

$i = \mathrm{III}$: With

$$B_{\mathrm{III}} = \begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix} \tag{18}$$

(Mulholland 1970) we get

$$y_{\mathrm{III}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathrm{III}} = \begin{pmatrix} \dot{u}/w \\ u \end{pmatrix} \tag{19}$$

and

$$M_{\mathrm{III}} = \begin{pmatrix} -\dot{w}/w & -w \\ w & 0 \end{pmatrix}. \tag{20}$$

These three parametrisations will not be used before § 4.

2.3. *Extraction of the trace part*

M from equation (5) is decomposed into

$$M = M_{\mathrm{T}} + M_{\mathrm{R}}, \tag{21}$$

where

$$M_{\mathrm{T}} = \tfrac{1}{2}(m_1 + m_4)\sigma_0 = \tfrac{1}{2}\sigma_0 \operatorname{Tr} M(t) \tag{22}$$

and

$$M_{\mathrm{R}} = \begin{pmatrix} (m_1 - m_4)/2 & m_2 \\ m_3 & -(m_1 - m_4)/2 \end{pmatrix}, \qquad \text{with } \operatorname{Tr} M_{\mathrm{R}} = 0. \tag{23}$$

With the factor decomposition of the integral matrix

$$Q = Q_{\mathrm{R}} Q_{\mathrm{T}}, \tag{24}$$

we get from equation (8) the two matrix DE

$$\dot{Q}_{\mathrm{T}} = (Q_{\mathrm{R}}^{-1} M_{\mathrm{T}} Q_{\mathrm{R}}) Q_{\mathrm{T}} = M_{\mathrm{T}} Q_{\mathrm{T}} = \tfrac{1}{2} Q_{\mathrm{T}} \operatorname{Tr} M(t), \qquad Q_{\mathrm{T}}(0) = \sigma_0 \tag{25}$$

and

$$\dot{Q}_{\mathrm{R}} = M_{\mathrm{R}} Q_{\mathrm{R}}, \qquad Q_{\mathrm{R}}(0) = \sigma_0. \tag{26}$$

According to theorem 1 of the Appendix, equation (25) can immediately be solved:

$$Q_{\mathrm{T}}(t) = \exp\left(\int_0^t M_{\mathrm{T}}(s)\, \mathrm{d}s\right) = \sigma_0 \exp\left(\tfrac{1}{2}\int_0^t \operatorname{Tr} M(s)\, \mathrm{d}s\right), \tag{27}$$

a multiple of the unit matrix.

2.4. *Extraction of the antisymmetric part*

Next we decompose M_{R}, equation (23), into

$$M_{\mathrm{R}} = M_{\mathrm{A}} + M_{\mathrm{S}}, \tag{28}$$

where the antisymmetric and the symmetric part are respectively

$$M_{\mathrm{A}} = \tfrac{1}{2}(M_{\mathrm{R}} - M_{\mathrm{R}}^{\mathrm{T}}) = \tfrac{1}{2}(m_2 - m_3)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \tfrac{1}{2}(m_2 - m_3) X_1 \tag{29}$$

(see Appendix) and

$$M_{\mathrm{S}} = \tfrac{1}{2}(M_{\mathrm{R}} + M_{\mathrm{R}}^{\mathrm{T}}) = \tfrac{1}{2}\begin{pmatrix} m_1 - m_4 & m_2 + m_3 \\ m_2 + m_3 & -(m_1 - m_4) \end{pmatrix}. \tag{30}$$

With the factor decomposition

$$Q_{\mathrm{R}} = Q_{\mathrm{A}} Q_{\mathrm{S}}, \tag{31}$$

we obtain from equation (26) the two DE

$$\dot{Q}_{\mathrm{A}} = M_{\mathrm{A}} Q_{\mathrm{A}}, \qquad Q_{\mathrm{A}}(0) = \sigma_0, \tag{32}$$

$$\dot{Q}_{\mathrm{S}} = T Q_{\mathrm{S}}, \qquad Q_{\mathrm{S}}(0) = \sigma_0, \tag{33}$$

with

$$T = Q_{\mathrm{A}}^{-1} M_{\mathrm{S}} Q_{\mathrm{A}}. \tag{34}$$

Equation (32) can again be solved according to theorem 1: with

$$f(t) = \tfrac{1}{2}\int_0^t (m_2(s) - m_3(s))\, \mathrm{d}s \tag{35}$$

we find

$$Q_A(t) = \exp(f(t)X_1) = \sigma_0 \cos f + X_1 \sin f = \begin{pmatrix} \cos f & \sin f \\ -\sin f & \cos f \end{pmatrix}, \tag{36}$$

a two-dimensional rotation matrix. Here equation (77) has been used. Thus from (34):

$$T = \tfrac{1}{2}\begin{pmatrix} (m_1 - m_4)\cos 2f - (m_2 + m_3)\sin 2f & (m_1 - m_4)\sin 2f + (m_2 + m_3)\cos 2f \\ (m_1 - m_4)\sin 2f + (m_2 + m_3)\cos 2f & -(m_1 - m_4)\cos 2f + (m_2 + m_3)\sin 2f \end{pmatrix}$$

$$= \begin{pmatrix} t_1 & t_2 \\ t_2 & -t_1 \end{pmatrix}. \tag{37}$$

Note that both the extraction of the trace part and the extraction of the antisymmetric part are Lyapunov transformations (with $B_i = Q_T$, Q_A respectively).

3. The method of decoupled exponentials

3.1. *Derivation of the Riccati* DE *for the exponents*

We use the fact that each 2×2 matrix with vanishing trace can be represented by a linear combination of the Pauli matrices σ_1, σ_2, σ_3, or by some linear combination X_1, X_2, X_3 of the basic matrices σ_i (see Appendix). Especially we have chosen

$$X_1 = i\sigma_2, \qquad X_2 = \sigma_3, \qquad X_3 = (\sigma_1 + i\sigma_2)/2 = \sigma_+. \tag{38}$$

Now the remaining DE (33) with T from equation (37) can be solved at least locally in the form

$$Q_S(t) = \exp\Big(\sum_i h_i(t)X_i\Big), \tag{39}$$

but in general not globally (Magnus 1954, Wichmann 1961). However, Wei and Norman (1963, 1964) and Wei (1963) have shown that the *ansatz*

$$Q_S(t) = \prod_{i=1}^{3} \exp(g_i(t)X_i), \qquad \text{where } g_i(0) = 0, \tag{40}$$

of decoupled exponentials is superior to the form (39); in the case of a 2×2 matrix, this solution holds globally if a suitable basis X_i is chosen. Besides (38) there are other combinations of the matrices σ_1, σ_2, σ_3 which will do it too (but not the original Pauli matrices, for example). One could have applied a corresponding factorisation immediately to the DE (8); however, we have preferred first to simplify the DE as far as possible. Explicitly, we have (cf equations (75), (76), (77))

$$Q_S(t) = e^{g_1 X_1}\, e^{g_2 X_2}\, e^{g_3 X_3}$$

$$= \begin{pmatrix} \cos g_1 & \sin g_1 \\ -\sin g_1 & \cos g_1 \end{pmatrix} \begin{pmatrix} \exp g_2 & 0 \\ 0 & \exp(-g_2) \end{pmatrix} \begin{pmatrix} 1 & g_3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos g_1 \exp g_2 & g_3 \cos g_1 \exp g_2 + \sin g_1 \exp(-g_2) \\ -\sin g_1 \exp g_2 & -g_3 \sin g_1 \exp g_2 + \cos g_1 \exp(-g_2) \end{pmatrix}. \tag{41}$$

Substitution into equation (33) yields a system of four equations, which we want to solve for the three derivatives $\dot{g}_i$. This system is consistent; the solution is

$$\dot{g}_1 = -(t_1 \sin 2g_1 + t_2 \cos 2g_1), \tag{42a}$$

$$\dot{g}_2 = t_1 \cos 2g_1 - t_2 \sin 2g_1, \tag{42b}$$

$$\dot{g}_3 = 2 \exp(-2g_2)(t_1 \sin 2g_1 + t_2 \cos 2g_1). \tag{42c}$$

Here one can see the usefulness of the basis X_i: (i) the system of four equations is solvable; and (ii) the system (42) is decoupled. A simple transformation

$$h = \tan g_1 \qquad \text{or } g_1 = \tan^{-1} h \tag{43}$$

transforms equation (42*a*) into a Riccati DE

$$\dot{h} = -2t_i h - t_2(1-h^2), \qquad \text{with } h(0) = 0. \tag{44}$$

Having solved equation (44), one knows g_1, and equations (42*b*, *c*) can be solved by simple quadratures. This is the best that can be expected: the reduction of the whole problem to the solution of the Riccati DE (44).

3.2. *Total solution*

We have, putting together equations (24) with (27), (31) with (36), and (41):

$$Q(t) = Q_R Q_T = Q_T Q_R = Q_T Q_A Q_S = Q_T \begin{pmatrix} \cos f & \sin f \\ -\sin f & \cos f \end{pmatrix} e^{g_1 X_1} e^{g_2 X_2} e^{g_3 X_3}$$
$$= Q_T \begin{pmatrix} \cos(f+g_1) \exp g_2 & \cos(f+g_1) g_3 \exp g_2 + \sin(f+g_1) \exp(-g_2) \\ -\sin(f+g_1) \exp g_2 & -\sin(f+g_1) g_3 \exp g_2 + \cos(f+g_1) \exp(-g_2) \end{pmatrix}. \tag{45}$$

We define new variables $\rho(t)$ and $\alpha(t)$ by (cf equation (7))

$$\rho \cos \alpha = y_{20} \exp(-g_2), \qquad \rho \sin \alpha = (y_{10} + g_3 y_{20}) \exp g_2; \tag{46}$$

therefore

$$\tan \alpha = (\tan \beta + g_3) \exp(2g_2), \tag{47a}$$

$$\rho^2 = y_{20}^2 \exp(-2g_2)/\cos^2 \alpha. \tag{47b}$$

Then, according to equation (6),

$$y(t) = \exp\left(\tfrac{1}{2} \int_0^t \operatorname{Tr} M(s)\, ds\right) \rho(t) \begin{pmatrix} \sin(f+g_1+\alpha) \\ \cos(f+g_1+\alpha) \end{pmatrix}. \tag{48}$$

This result was to be expected: for $w^2(t) = \omega_0^2 =$ constant, the solution of equation (1) is

$$u(t) = a \cos(\omega_0 t + \psi).$$

A function $w = w(t)$ changes, in general, the amplitude as well as the phase into functions depending on t.

4. The WKB approximation

We start with the second parametrisation, equations (16), (17). Since $m_2 = m_3$, we have exactly $f = 0$, equation (35), and $Q_A(t) = \sigma_0$. Van Kampen (1967) has shown that one can get the WKB approximation by neglecting the non-diagonal terms in (17). Therefore

$$M_{II} \approx \begin{pmatrix} iw - \dot{w}/2w & 0 \\ 0 & -iw - \dot{w}/2w \end{pmatrix}, \tag{49}$$

and from equation (37)

$$T \approx iw \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = iw(t)\sigma_3. \tag{50}$$

According to theorem 1 of the Appendix we find from (33) with (75)

$$Q_S = e^{ir(t)\sigma_3} = \begin{pmatrix} \exp(ir) & 0 \\ 0 & \exp(-ir) \end{pmatrix} = e^{g_2 X_2} \qquad (g_1 = g_3 = 0), \tag{51}$$

where

$$r(t) = \int_0^t w(s)\, ds. \tag{52}$$

Thus, with (45), for the second form of parametrisation in the WKB approximation

$$Q = Q_{II} = \left(\frac{w(0)}{w(t)}\right)^{1/2} \begin{pmatrix} \exp(ir) & 0 \\ 0 & \exp(-ir) \end{pmatrix}. \tag{53}$$

In the first form of parametrisation we obtain from (14)

$$Q_I = B_{II}(t) Q_{II}(t) B_{II}^{-1}(0), \tag{54}$$

where B_{II} is given by (15). Explicitly

$$Q_I(t) = (w(0)w(t))^{-1/2} \begin{pmatrix} w(t)\cos r & -w(t)w(0)\sin r \\ \sin r & w(0)\cos r \end{pmatrix}, \tag{55}$$

and with equation (10)

$$u(t) = (w(0)w(t))^{-1/2} \dot{u}(0) \sin r + (w(0)/w(t))^{1/2} u(0) \cos r. \tag{56}$$

For the third form of parametrisation we have from (14)

$$Q_{III}(t) = B_{III}^{-1}(t) Q_I(t) B_{III}(0), \tag{57}$$

where B_{III} is given by (18). Explicitly

$$Q_{III}(t) = (w(0)/w(t))^{1/2} \begin{pmatrix} \cos r & -\sin r \\ \sin r & \cos r \end{pmatrix} \tag{58}$$

is the expression of the WKB approximation for the third form of parametrisation.

On the other hand, we have directly from M_{III} in (20) $m_2 = -w(t) = -m_3$; therefore, according to (35)

$$f(t) = -\int_0^t w(s)\, ds = -r(t), \tag{59}$$

and with equation (36)

$$Q_A(t) = \begin{pmatrix} \cos r & -\sin r \\ \sin r & \cos r \end{pmatrix}. \tag{60}$$

From equation (27)

$$Q_T(t) = (w(0)/w(t))^{1/2}\sigma_0. \tag{61}$$

Finally, according to (37), for small values of $\dot{w}/2w$

$$T = -\frac{\dot{w}}{2w}\begin{pmatrix} \cos 2r & -\sin 2r \\ -\sin 2r & -\cos 2r \end{pmatrix} \approx 0. \tag{62}$$

Then $Q_S \approx \text{constant} = \sigma_0$, and $Q = Q_{III}$ agrees with (58). But $Q_S = \sigma_0$ implies $g_1 = g_2 = g_3 = 0$. Therefore each set $\{g_1, g_2, g_3\} \neq \{0, 0, 0\}$ gives, in the third form of parametrisation, an improvement compared with the WKB approximation. Such a set will be calculated in the next section.

5. Approximate solution

We assume in equation (37) that t_1 and t_2 both contain a small factor ϵ:

$$t_1 = \epsilon T_1, \qquad t_2 = \epsilon T_2 \tag{63}$$

(other cases can be discussed correspondingly). Then the series expansion

$$h(t) = h_0(t) + \epsilon h_1(t) + \epsilon^2 h_2(t) + \ldots, \tag{64}$$

with

$$h_i(0) = 0 \qquad (i = 0, 1, \ldots),$$

is substituted into (44). Equating coefficients of equal powers yields

$$\begin{aligned} &\dot{h}_0 = 0 && \text{or } h_0 = 0, \\ &\dot{h}_1 = -T_2 && \text{or } h_1 = -\int_0^t T_2(s)\,\mathrm{d}s, \\ &\dot{h}_2 = -2h_1 T_1 && \text{or } h_2 = 2\int_0^t \mathrm{d}s\, T_1(s)\int_0^s \mathrm{d}s'\, T_2(s'), \end{aligned} \tag{65}$$

and so on. Therefore

$$\begin{aligned} h(t) &= -\epsilon\int_0^t T_2(s)\,\mathrm{d}s + 2\epsilon^2\int_0^t \mathrm{d}s\, T_1\int_0^s \mathrm{d}s'\, T_2 + \ldots \\ &= -\int_0^t t_2(s)\,\mathrm{d}s + 2\int_0^t \mathrm{d}s\, t_1(s)\int_0^s \mathrm{d}s'\, t_2(s') + \ldots. \end{aligned}$$

In lowest order in ϵ we obtain from equation (43)

$$g_1 \approx -\int_0^t t_2(s)\,\mathrm{d}s \sim \epsilon, \tag{66a}$$

and from (42*b*, *c*)

$$g_2 \approx \int_0^t t_1(s)\,\mathrm{d}s \sim \epsilon, \tag{66b}$$

$$g_3 \approx 2\int_0^t t_2(s)\,\mathrm{d}s \sim \epsilon. \tag{66c}$$

The *ansatz*

$$\alpha = \beta + \gamma = \beta + \epsilon\Gamma, \tag{67}$$

where β is given by equation (7), yields $\tan\alpha \approx \tan\beta + \gamma(1+\tan^2\beta)$. Comparing with (47*a*) we get

$$\gamma \approx 2y_{20}\left(y_{10}\int_0^t t_1(s)\,\mathrm{d}s + y_{20}\int_0^t t_2(s)\,\mathrm{d}s\right)\Big/(y_{10}^2+y_{20}^2) \sim \epsilon. \tag{68}$$

Then the total phase in equation (48) reads

$$\begin{aligned}\phi &= f + g_1 + \alpha = (f+\beta) + (g_1+\gamma) = (f+\beta) + \epsilon \ldots \\ &= (f+\beta) + \sin 2\beta \int_0^t t_1(s)\,\mathrm{d}s + \cos 2\beta \int_0^t t_2(s)\,\mathrm{d}s.\end{aligned} \tag{69}$$

The main contribution comes from the antisymmetric matrix $M_\mathrm{A}(f)$ and from the initial conditions. The small contributions come from the symmetric, traceless matrix M_S (or T). The matrix M_T with non-vanishing trace does not influence the phase at all.

For the factor $\rho(t)$ in the amplitude of equation (48) we get now from (47*b*)

$$\rho(t) = \rho_0\left(1 - \cos 2\beta \int_0^t t_1(s)\,\mathrm{d}s + \sin 2\beta \int_0^t t_2(s)\,\mathrm{d}s\right). \tag{70}$$

Contributions to ρ come from the initial conditions (ρ_0 and β) and from the symmetric, traceless matrix M_S. Finally, the matrix M_T with non-vanishing trace also contributes to the total amplitude.

Equations (69) and (70) can be simplified by the definitions

$$\rho_1(t)\cos\delta(t) = \int_0^t t_1(s)\,\mathrm{d}s, \qquad \rho_1(t)\sin\delta(t) = \int_0^t t_2(s)\,\mathrm{d}s, \qquad \rho_1 \sim \epsilon. \tag{71}$$

Then

$$\phi = (f+\beta) + \rho_1(t)\sin(2\beta + \delta(t)), \qquad \rho(t) = \rho_0[1 - \rho_1(t)\cos(2\beta + \delta(t))]. \tag{72}$$

Appendix. Some elementary matrix algebra

Theorem 1. Given the matrix DE $\dot{Z}(t) = K(t)Z(t)$ with $Z(0) = I$ and $K(t)$ analytic in I_t, then a solution of the form

$$Z(t) = \exp\left(\int_0^t K(s)\,\mathrm{d}s\right)$$

exists, if and only if the commutator

$$\left[K(t), \int_0^t K(s)\,\mathrm{d}s\right] = 0$$

for all $t \in I_t$. For a proof see, for example, Martin (1968).

For the Pauli matrices we use the following representation:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = X_2. \tag{73}$$

Then

$$X_1 = \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \sigma_+ = (\sigma_1 + \mathrm{i}\sigma_2)/2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{74}$$

We need the exponentials (defined as $\mathrm{e}^A = \Sigma_{n=0}^{\infty} A^n/n!$)

$$\mathrm{e}^{\alpha X_2} = \exp\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} = \begin{pmatrix} \exp\alpha & 0 \\ 0 & \exp(-\alpha) \end{pmatrix}, \tag{75}$$

$$\mathrm{e}^{\alpha X_3} = \sigma_0 + \alpha X_3 = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \qquad \text{since } X_3^2 = 0. \tag{76}$$

$\mathrm{e}^{\alpha X_1}$ can also be calculated by a series expansion, or, with

$$P = \begin{pmatrix} 1 & -1 \\ \mathrm{i} & \mathrm{i} \end{pmatrix} \qquad \text{and } D = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix},$$

as follows:

$$\begin{aligned} \mathrm{e}^{\alpha X_1} &= \exp(\alpha PDP^{-1}) = P\,\mathrm{e}^{\alpha D}P^{-1} = P\begin{pmatrix} \exp(\mathrm{i}\alpha) & 0 \\ 0 & \exp(-\mathrm{i}\alpha) \end{pmatrix}P^{-1} \\ &= \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \sigma_0\cos\alpha + X_1\sin\alpha. \end{aligned} \tag{77}$$

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